Admissions Testing Service

STEP Solutions 2014

Mathematics

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Test

STEP 3 2014 Hints and Solutions

1. The stem results are obtained through algebraic expansion and equating coefficients. Using the expression $(1+ax)(1+bx)(1+cx)$ for $1+qx^2+rx^3$, manipulating the logarithm of the product, and the series expansions for expressions like $\ln(1+ax)$ yields the displayed result. In parts (ii), (iii), and (iv), it is simplest to find $S_2=-q$, $S_3=r$, $S_5=-qr$, $S_7=q^2r$, and $S_9=\frac{r^3}{3}-q^3r$ by expanding the series for $\ln\left(1+(qx^2+rx^3)\right)$, and choosing a counter-example, selecting a, b and c so that $r\neq 0$.

2. The first part is solved using the given method, the formula $\cosh 2x=2\cosh^2 x-1$, and then employing partial fractions or the standard form quoted in the formula book. The second part requires the substitution, $u=\sinh x$, the formula $\cosh 2x=1+2\sinh^2 x$, and a standard form to give $\frac{\sqrt{2}}{2}\tan^{-1}\sqrt{2}\,u+c$. The third part can be approached by making the substitution $u=e^x$ and division of the resulting fraction in the numerator and denominator by e^{2x} to give half the difference of the integrals in the first two parts. Alternatively, a similar style of working with the substitution $u=e^{-x}$ results in a sum instead of a difference.

3. (i) Given that the shortest distance between the line and the parabola will be zero if they meet, investigating the solution of the equations simultaneously , and the discriminant of the resulting quadratic equation, the first result of the question is the case that they do not meet. The closest approach is the perpendicular distance of the point on the parabola where the tangent is parallel to the line, so using the standard parametric form, it is the perpendicular distance of $\left(\frac{a}{m^2},\frac{2a}{m}\right)$ from $=mx+c$, giving the required result with care being taken over the sign of the numerator bearing in mind the inequalities.

(ii) The shortest distance of a point on the axis from the parabola, is either the distance from the vertex to the point, or the distance along one of the normals (which are symmetrically situated) which is not the axis. If the normal at $(at^2,2at)$ passes through $(p,0)$, then $=2a+at^2$. From this it can be simply shown that shortest distance is p if $\frac{p}{a}<2$, and is $2\sqrt{a(p-a)}$ if $\frac{p}{a}\geq 2$.

Then for the circle, the results follow simply, that the shortest distance will be $p-b$ if $p>b$, and 0 otherwise if $\frac{p}{a}<2$, and $2\sqrt{a(p-a)}-b$ if $4a(p-a)>b^2$ or 0 otherwise if $\frac{p}{a}\geq 2$.

4. Expanding the bracket in the integral I_1, and employing $\sec^2 x=1+\tan^2 x$ yields I plus the integral of a perfect differential which can be evaluated simply. For $I=0$, $y'+y\tan x=0$ over the interval which can be solved using an integrating factor and then the condition $y=0, x=1$ enables the arbitrary constant to be evaluated giving the required result. In part (ii), similar working can be undertaken with the integral which is to be considered, given $b=a$. The argument requires no discontinuity in the interval so $a<\frac{\pi}{2}$. The function $y=\cos ax$ can be shown to meet the requirement.

5. ABCD is a parallelogram if and only if $\overrightarrow{AB} = \overrightarrow{DC}$ which yields the required result. To be a square as well, angle $ABC = 90^o$, and $|AB| = |BC|$, so $c - b = i(b - a)$. Treating the two results as simultaneous equations to be solved for a and c in terms of b and d, the second result of the stem can be shown with reversible logic. For part (i) the same logic can be used for PXQ as just used for ABC. From the stem, XYZT is a square if and only if $i(x - z) = y - t$, and

$x + z = y + t$ and given the working for X in part (i), these can be shown to be true treating Y, Z, and T similarly

6. Starting from $f''(t) > 0$ for $0 < t < x_0 \implies \int_0^{t_0} f''(t)\,dt > 0$ where $0 < t_0 < x_0$, with the given conditions yields $f'(t_0) > 0$, and then repeating the argument with $f'(t)$ instead gives $f(t) > 0$. Choosing $f(x) = 1 - \cos x \cosh x$ and applying the applying the stem of the question for $0 < x < \pi$, gives the required inequality for $0 < x < \pi/2$ in particular. For part (ii), choosing $g(x) = x^2 - \sin x \sinh x$ (in which case $g''(x) = 2f(x)$, where $f(x)$ was the suggested choice for part (i)) and $h(x) = \sin x \cosh x - x$ provide the desired results once care is taken with positivity of functions over the required interval when dividing inequalities.

7. Part (i), the intersecting chords theorem, is basic bookwork relying on angle properties in circles to establish similar triangles and hence the result. Part (ii) can be obtained by considering that Q lies on P_1P_3 and so $\boldsymbol{q} = \boldsymbol{p_1} + \lambda(\boldsymbol{p_3} - \boldsymbol{p_1})$, that Q also lies on P_2P_4 producing a similar result and then equating these two expressions, finally rearranging to give (*). Assuming that $a_1 + a_3 = 0$ and using (*) leads to $a_1(\boldsymbol{p_1} - \boldsymbol{p_3}) = a_2(\boldsymbol{p_4} - \boldsymbol{p_2})$ which, in view of the distinctness of the four points P and the intersection of P_1P_3 and P_2P_4 at Q , leads to the contradiction $a_1 = a_2 = a_3 = a_4 = 0$. Re-writing $\frac{a_1\boldsymbol{p_1}+a_3\boldsymbol{p_3}}{a_1+a_3}$ as $\boldsymbol{p_1} + \frac{a_3(\boldsymbol{p_3}-\boldsymbol{p_1})}{a_1+a_3}$ and similarly, using (*), as $\frac{a_2\boldsymbol{p_2}+a_4\boldsymbol{p_4}}{a_2+a_4}$ and re-writing, the expression can be shown to be the position vector of Q .The final result comes from applying (i) using the information just gained and calculating both expressions by taking scalar products of the vectors whose magnitudes are quoted in (i).

8. The initial result is obtained by extending the given inequality so that each term of the sum is compared with $f(k^n)$ and $f(k^{n+1})$. Part (i) is obtained using the stem, the given function, $= 2$, and summing the sums. The deduction relies on considering the lower limit of the sum. The same approach applies to part (ii), with the new function given and considering the upper limit which is obtained as a geometric progression. Counting the number of elements of $S(1000)$ gives the method for obtaining $\sigma(n)$ using the same function as part (i) except $f(r) = 0$ if r has one or more 2 s in its decimal representation and with $k = 10$, again with the sum of a geometric progression. The final result is particularly attractive, demonstrating how few terms need to be removed from the non-convergent harmonic progression (of part (i)) in order to produce a convergent sequence.

9. $\mathbf{v} = \frac{1-\mathrm{e}^{-\mathrm{k}t}}{\mathrm{k}}\mathbf{g} + \mathrm{e}^{-\mathrm{k}t}\mathbf{u}$ and a further differentiation yields $\mathrm{m}\mathbf{a} = \mathrm{m}\mathbf{g} - \mathrm{mk}\mathbf{v}$ **.** Using $\mathrm{r.j} = 0$

obtains the first displayed result after re-arrangement, as does $\tan\beta = \frac{\mathbf{-v.j}}{\mathbf{v.i}}$ the second.

$\tan\beta - \tan\alpha$ can be shown to be $\frac{2\mathrm{g}}{\mathrm{uk}\cos\alpha(1-\mathrm{e}^{-\mathrm{kT}})}(\sinh\mathrm{kT} - \mathrm{kT})$ which leads to the two final inequalities.

10. The first result is obtained by considering Newton's second law applied to the mass X under the tension in PX and the thrust of XY. $m\frac{d^2y}{dt^2} = -\frac{\lambda(x+2y)}{a}$ is similarly obtained considering Y. Subtracting the two equations gives a SHM second order differential equation for $-y$, and adding them gives similar for $x + y$. Solving these using initial conditions give $x - y = \frac{1}{2}a\cos\omega t$ and $x + y = -\frac{1}{2}a\cos\sqrt{3}\omega t$. The final result is particularly elegant, and possibly a little surprising that a conservative oscillating system does not return to its starting position. Treating the previous two results as simultaneous equations for x and y, and solving $y = -\frac{1}{2}a$, yields $1 = \cos\sqrt{3}\omega t$ and $1 = \cos\omega t$, so that $\sqrt{3}\omega t = 2n\pi$ and $\omega t = 2m\pi$ for non-zero integers n and m, yielding the contradiction $\sqrt{3} = \frac{n}{m}$.

11. Resolving vertically and horizontally, and solving the resulting simultaneous equations and then tidying up the trigonometric expressions yields $T_A = m\frac{(g\sin\beta+\omega^2 x\sin\alpha\cos\beta)}{\sin(\alpha+\beta)}$ and $T_B = m\frac{(\omega^2 x\sin\alpha\cos\alpha - g\sin\alpha)}{\sin(\alpha+\beta)}$. Trivially, the former is positive, but the same condition applied to the latter, given that its denominator and the common factor of the numerator can be shown to be positive, yields the first required inequality. The geometric inequality could be proved, as candidates tended to, by use of the cosine rule and then completing the square to obtain $(x - h\cos\alpha)^2 = d^2 - h^2 + h^2\cos^2\alpha$. However, use of the sine rule and the maximum of the sine function, or the shortest distance of B from AP gives $d \geq h\sin\alpha$, which along with $\cos^2\alpha + \sin^2\alpha = 1$, give the required inequality. In the particular case, it can be shown that $T_A = \frac{mg}{\cos\alpha}$ and the knowledge of the unattainable maximum value of the cosine function along with the geometric inequality previously obtained leads to the final inequalities. The geometry is that the strings are perpendicular, which can be appreciated by considering the equality case of $d \geq h\sin\alpha$.

12. The first result, $y_m = e^{x_m}$, is obtained merely by considering probabilities, and the given pdf of Y can be obtained by standard techniques or by consideration of changing the variable in the integral of the pdf of X. The mode result relies on differentiation of the pdf of Y equated to zero to give a stationary value. The explanation in part (iii) is simply that the required integral is merely that of the pdf of a Normal variable with mean $\mu + \sigma^2$. The expectation of Y is obtained in the standard manner, using an integral and the pdf of Y, and then a change of variable, in which exponential terms can be combined so as to use the explained result having completed the square in the exponent. Using the three previous parts gives $\lambda = e^{\mu-\sigma^2}$, $y_m = e^{x_m} = e^{\mu}$ because X is symmetric, and, as stated,

$E(Y) = e^{\mu+\frac{1}{2}\sigma^2}$, hence satisfying part (iv).

13. The first result is a trivial application of the definition of a probability generating function, and the second similarly. In order to obtain the first printed result in part (iii), it is necessary to obtain a similar result to those in parts (i) and (ii) giving $tG(t)$ as the score is one higher and then applying the conditionality of the probabilities of these three results which is done

by considering the probability of a score n in the three cases to give the coefficient of t^n. Re-arranging the formula for $G(t)$, either differentiation or the binomial theorem can be used to find the required probability formula. Finding $\mu = G'(1) = c/a$ and the knowledge that $a + b + c = 1$ enables the result of part (iii) to be rearranged to that of part (iv).